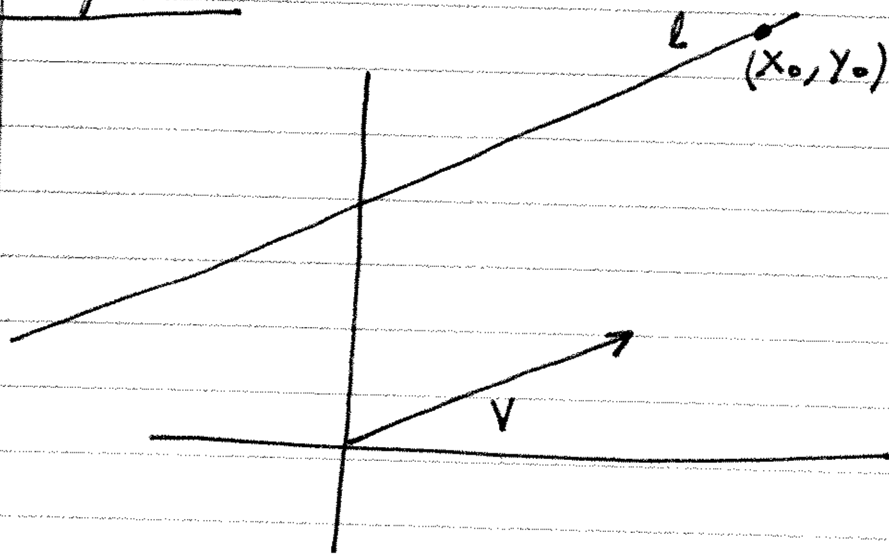
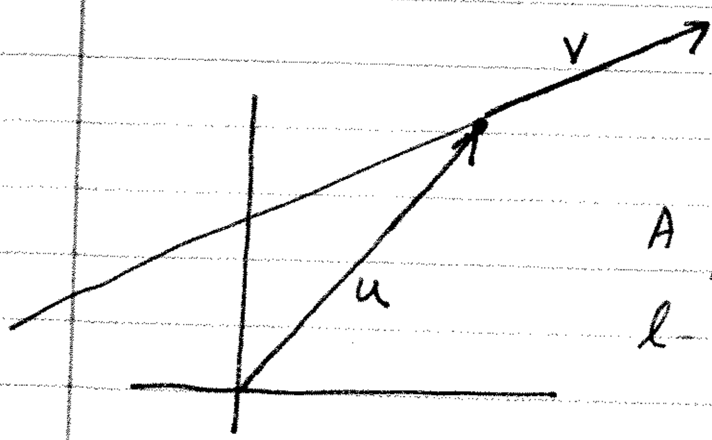
# Lines and planes in $\mathbb{R}^n$

## Example 1




Find $l$ which passes through $(x_0, y_0)$ and is parallel to vector $v = (\alpha, \beta)$.




Define $u = (x_0, y_0)$

A parameterization of $l$ is given by

$$l : \{ u + \lambda v : \lambda \in \mathbb{R} \}.$$

$$u + \lambda v = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x_0 + \lambda\alpha \\ y_0 + \lambda\beta \end{pmatrix}$$

write $x = x_0 + \lambda\alpha$
$y = y_0 + \lambda\beta$

$$\boxed{\frac{x - x_0}{\alpha} = \frac{y - y_0}{\beta}.}$$

Required Equation of the line

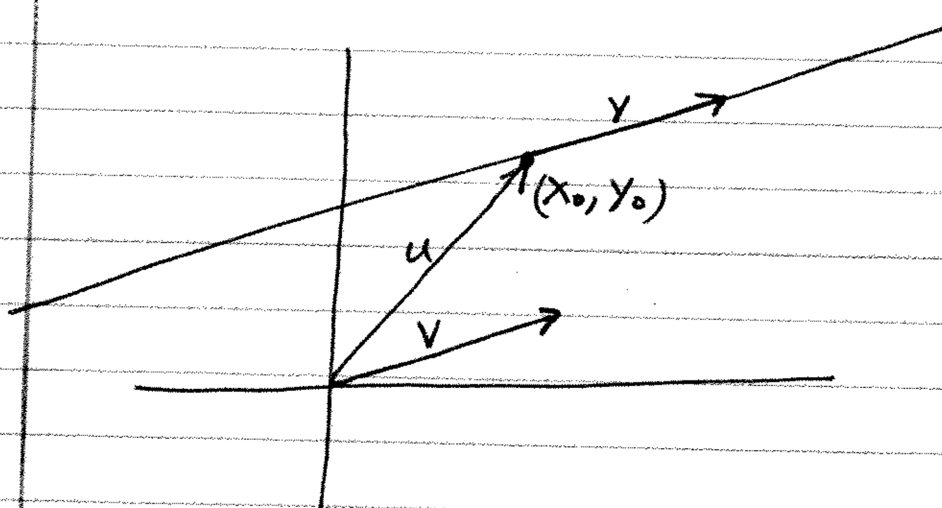
— x —

## Example 2

Let

$$a(x-x_0)+b(y-y_0)=0$$

be the cartesian equation of a line $l$ in $\mathbb{R}^2$, find a parameterization.




The line $l$ passes through $(x_0, y_0)$, hence

$$u = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Slope of the line is $-\frac{a}{b}$, hence we can choose

$$v = \begin{pmatrix} b \\ -a \end{pmatrix}$$

$u + \lambda v$

$$= \left\{ \begin{pmatrix} x_0 + \lambda b \\ y_0 - \lambda a \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

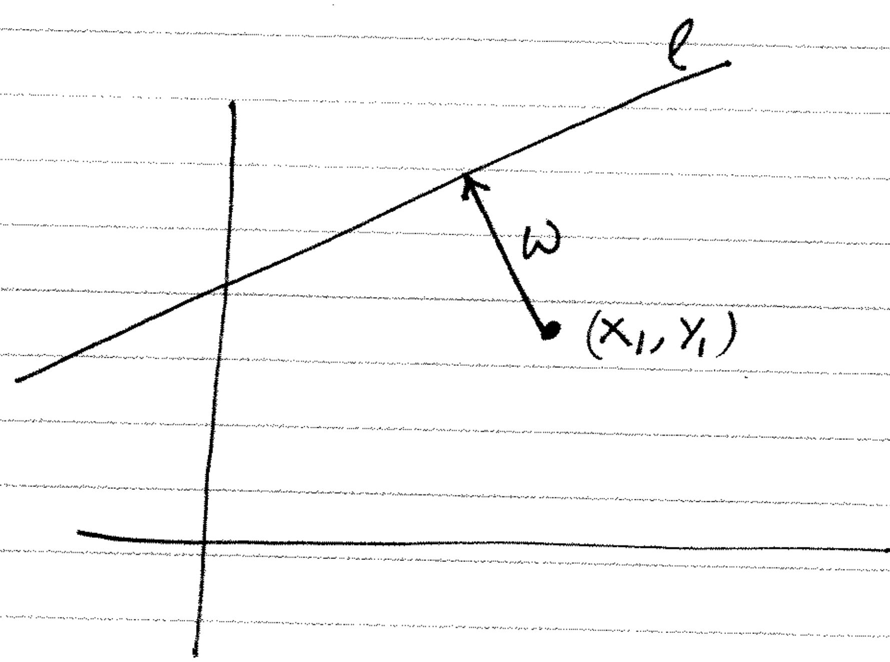
required parameterization.

Example 3

Find shortest distance between line

$$a(x-x_0)+b(y-y_0)=0$$

and point $(x_1, y_1)$.




From the parameterization of $l$ we have $\begin{pmatrix} x_0+\lambda b \\ y_0-\lambda a \end{pmatrix}$ is any pt. on $l$.

We need to choose $\lambda$:

$$\begin{pmatrix} x_0 + \lambda b \\ y_0 - \lambda a \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

is perpendicular to $l$, i.e.

$$\begin{pmatrix} (x_0 - x_1) + \lambda b \\ (y_0 - y_1) - \lambda a \end{pmatrix} \cdot \begin{pmatrix} b \\ -a \end{pmatrix} = 0$$

$$\Rightarrow b(x_0 - x_1) - a(y_0 - y_1) + \lambda(a^2 + b^2) = 0$$

$$\Rightarrow \lambda = \frac{a(y_0 - y_1) - b(x_0 - x_1)}{a^2 + b^2}$$

If $d$ is the magnitude of $\omega$, we have

$$d^2 = \left[(x_0 - x_1) + \lambda b\right]^2 + \left[(y_0 - y_1) - \lambda a\right]^2$$

$$= \left\{ \frac{a}{a^2+b^2} \left[ a(x_0 - x_1) + b(y_0 - y_1) \right] \right\}^2$$

$$+ \left\{ \frac{b}{a^2+b^2} \left[ a(x_0 - x_1) + b(y_0 - y_1) \right] \right\}^2$$

$$= \frac{1}{a^2+b^2} \left[ a(x_0 - x_1) + b(y_0 - y_1) \right]^2$$

$$d = \frac{|a(x_0 - x_1) + b(y_0 - y_1)|}{\sqrt{a^2 + b^2}}$$

Remark:

If $l$ is given by eq$^{n}$

$$ax + by + c = 0$$

and it passes through $(x_0, y_0)$ i.e if

$$a x_0 + b y_0 + c = 0$$

we have

$$d = \frac{|a x_1 + b y_1 + c|}{\sqrt{a^2 + b^2}}$$

## Example 4

Find equation of a line $l$ perpendicular to vector $\begin{pmatrix} a \\ b \end{pmatrix}$ through the point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Slope of $\begin{pmatrix} a \\ b \end{pmatrix}$ is $\frac{b}{a}$. Hence slope $m$ of the line is such that

$$m \frac{b}{a} = -1 \Rightarrow m = -\frac{a}{b}$$

$\therefore$ The line has equation.

$$y = mx + c$$

$$= -\frac{a}{b} x + c$$

$$\Rightarrow ax + by = C$$

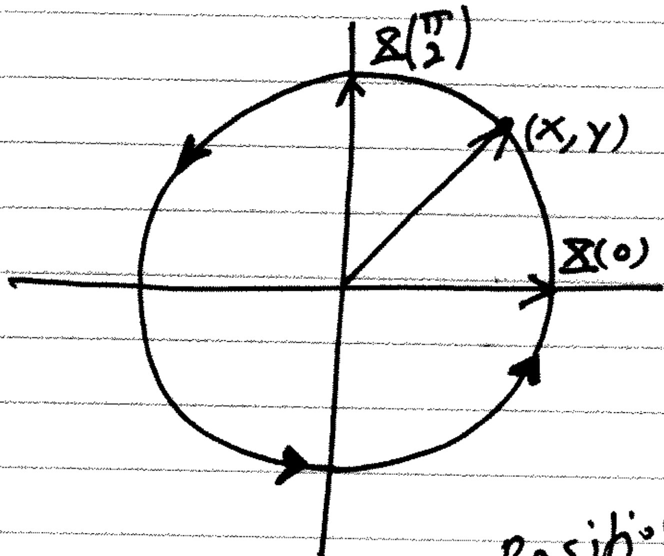
Since the line passes through $(x_0, y_0)$ we have

$$a(x_0) + b(y_0) = c$$

$$\Rightarrow a(x - x_0) + b(y - y_0) = 0$$

is the required line.

## Example 5




$$\mathbf{X} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

position vector of a particle at time $t$

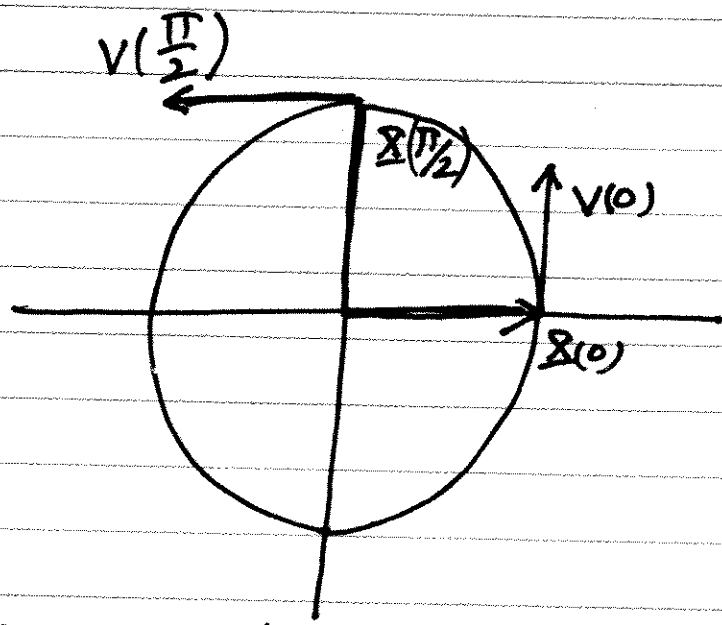
$$\mathbf{X}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathbf{X}\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Velocity

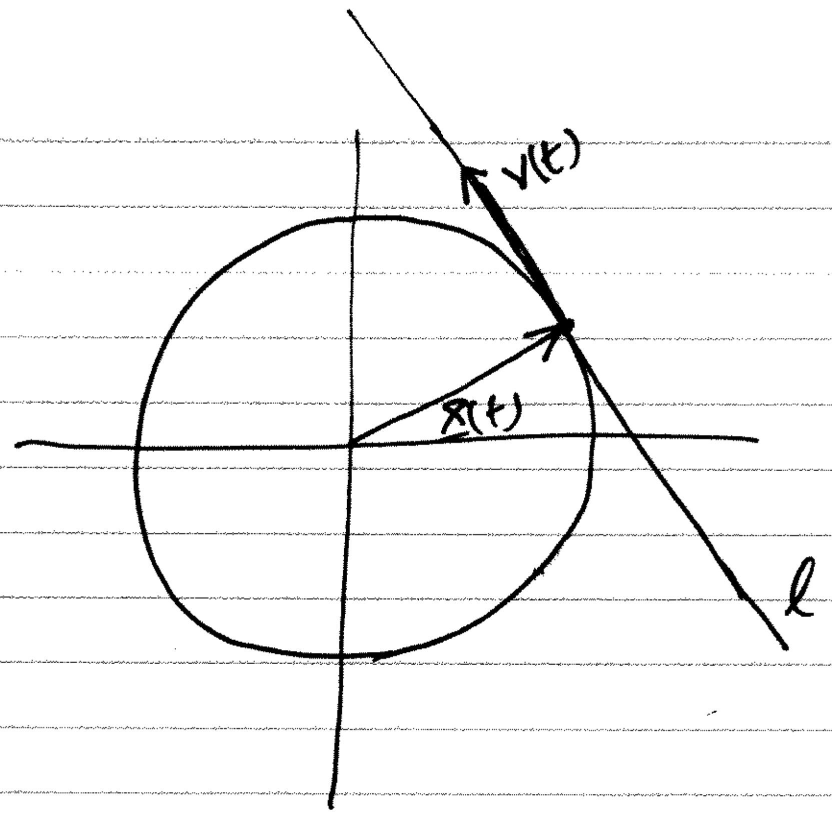
$$v(t) = \dot{\underline{x}}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$$v(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v\left(\frac{\pi}{2}\right) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$




Each velocity vector is a vector on the tangent line.




$$\underline{x}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad v(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

$\ell$ is parameterized by

$$\left\{ \begin{pmatrix} \cos t - \sin t\,\lambda \\ \sin t + \cos t\,\lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

Eq$^{\underline{n}}$: $\dfrac{x - \cos t}{-\sin t} = \dfrac{y - \sin t}{\cos t}$.

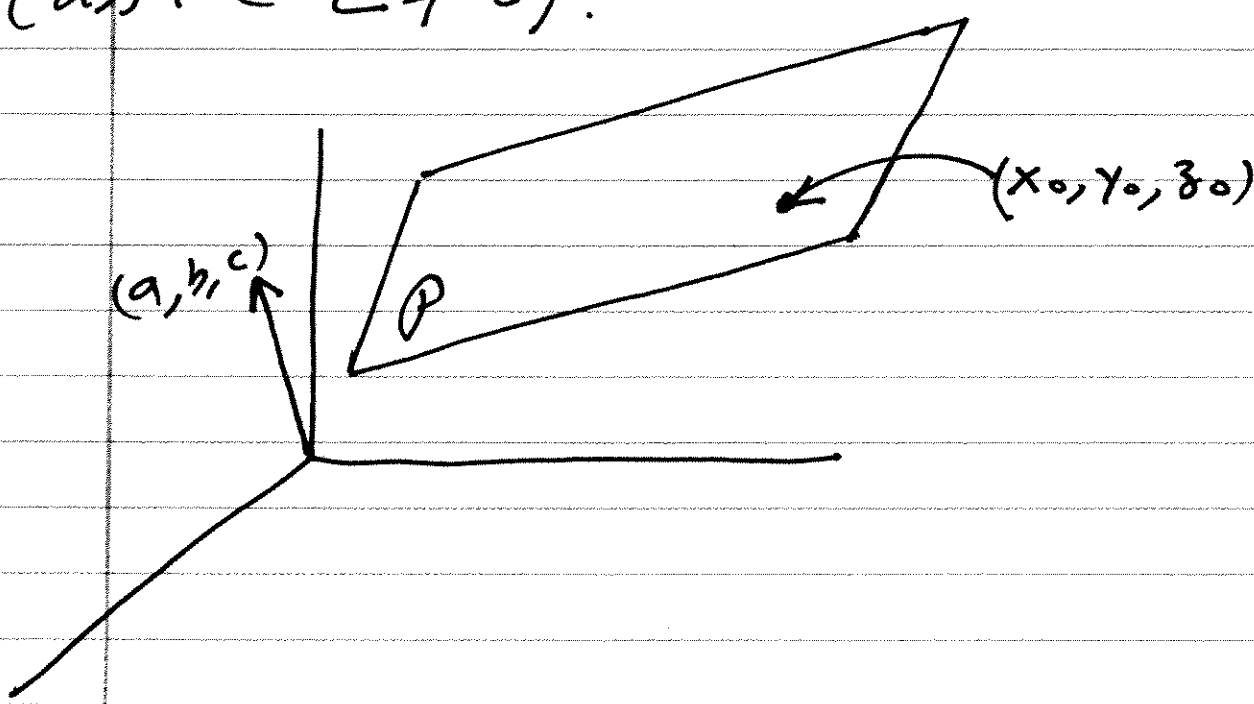
$\Rightarrow \cos t (x - \cos t) + \sin t (y - \sin t) = 0$

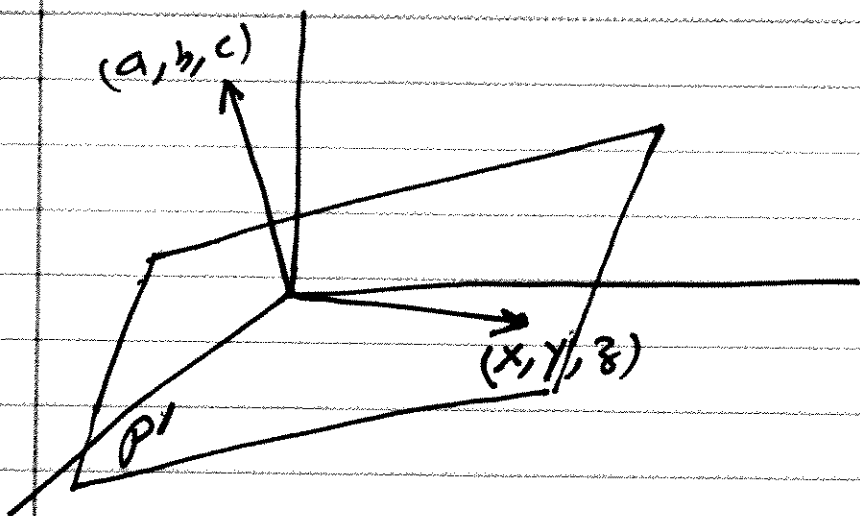
$\Rightarrow \cos t\, x + \sin t\, y = 1.$

Let us now talk about planes.

Example 6

Find $P$ which passes through the vector $(x_0, y_0, z_0)$ in $\mathbb{R}^3$ and is perpendicular to $(a, b, c)$. (assume $c \neq 0$).






Let us translate parallely the plane $P$ to a plane $P'$ which passes through the origin. $(a, b, c)$ is still perpendicular to $P'$.

Let $(x, y, z)$ be any point on $P'$. It follows that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \underbrace{ax + by + cz = 0}_{\text{Eq}^{\underline{n}} \text{ of } P'.}$$

Eqⁿ of $P$ is therefore going to be

$$ax + by + cz = d$$

for some choice of $d$ : $P$ passes through the point $(x_0, y_0, z_0)$.

Hence

$$d = ax_0 + by_0 + cz_0$$

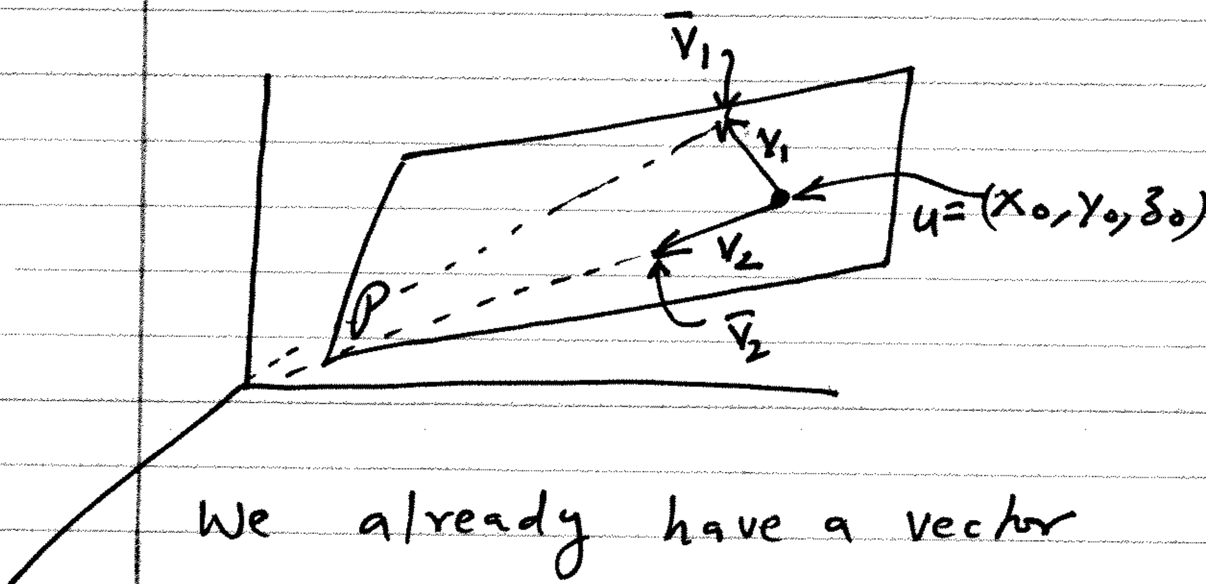
and we have

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

Eqⁿ of $P$.

## Example 7

Find a parametric description of $P$ in example 6




We already have a vector

$$u = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$ on the plane.

We want two additional vectors $\overline{v_1}, \overline{v_2}$ in the plane and calculate

$$v_1 = \overline{v_1} - u$$
$$v_2 = \overline{v_2} - u.$$

To calculate $\overline{V_1}$ choose

$x = x_0$

and we have

$by + cz = by_0 + cz_0$.

we choose

$y = y_0 - c, \quad z = z_0 + b$.

$$\overline{V_1} = \begin{pmatrix} x_0 \\ y_0 - c \\ z_0 + b \end{pmatrix}$$

Likewise to choose $\overline{V_2}$ we write

$y = y_0$

$ax + cz = ax_0 + cz_0$

and obtain.

$x = x_0 - c, \quad z = z_0 + a$.

$$\overline{V_2} = \begin{pmatrix} x_0 - c \\ y_0 \\ z_0 + a \end{pmatrix}$$

We write

$$v_1 = \bar{v}_1 - u = \begin{pmatrix} 0 \\ -c \\ b \end{pmatrix}$$

$$v_2 = \bar{v}_2 - u = \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}$$

Required parameterization of $P$ is

$$\{u + \lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}$$

$$= \left\{ \begin{pmatrix} x_0 - c\lambda_2 \\ y_0 - c\lambda_1 \\ z_0 + b\lambda_1 + a\lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

## Example 8

Find a parametric description of $P'$ in Example 6.

$\because P'$ is obtained by parallely translating $P$ and since $P$ is parameterized as.

$$\{u + \lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}$$

it would follow that $P'$ is parameterised as

$$\{\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

$$= \left\{ \begin{pmatrix} -c\lambda_2 \\ -c\lambda_1 \\ b\lambda_1 + a\lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

Remark:

$P'$ is the span of vectors

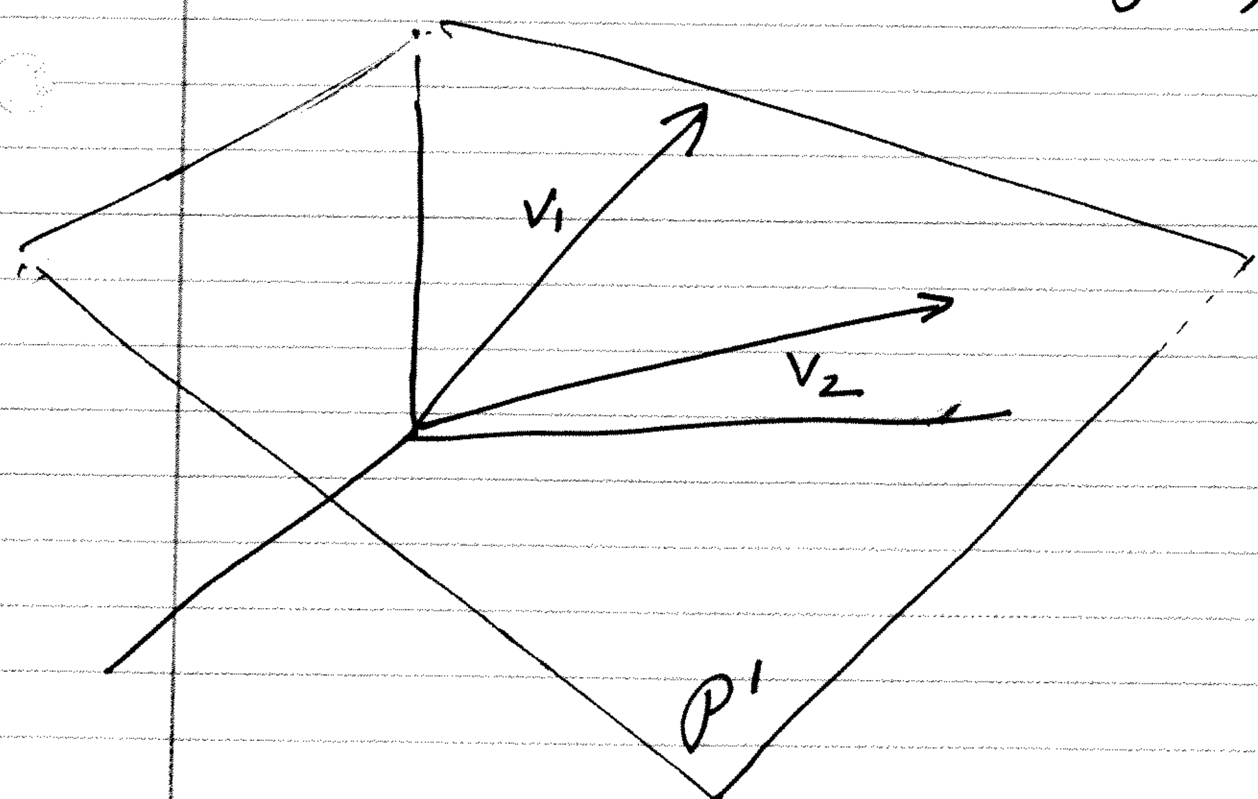
$v_1$ & $v_2$ where

$$v_1 = \begin{pmatrix} 0 \\ -c \\ b \end{pmatrix}, \quad v_2 = \begin{pmatrix} -c \\ 0 \\ a \end{pmatrix}$$

$$P' = [v_1, v_2].$$

Let us try to summarize what we have got so far.

If $V_1$ & $V_2$ are two non-zero vectors pointing in different directions in $\mathbb{R}^3$ (see Figure).




Then $[V_1, V_2]$ defines span of the two vectors $V_1$ & $V_2$.

- span of two such vectors $v_1$ & $v_2$ is a plane in $\mathbb{R}^3$ which passes through the origin.

Technically we say.

$[v_1\ v_2]$, the span of $v_1$ & $v_2$ is a homogeneous 2 plane in $\mathbb{R}^3$.

We can parametrize this plane as

$$\{\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

Example 9:

If $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$

$$P' = [v_1, v_2] = \left\{ \begin{pmatrix} \lambda_1 + 3\lambda_2 \\ 5\lambda_2 \\ \lambda_1 - \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

It is a homogeneous 2-plane in $\mathbb{R}^3$ spanned by $v_1$ & $v_2$.

We write down the equation of the plane as follows.

Write
$$\begin{aligned} x &= \lambda_1 + 3\lambda_2 \\ y &= 5\lambda_2 \\ z &= \lambda_1 - \lambda_2 \end{aligned}$$
and eliminate the parameters $\lambda_1, \lambda_2$

$$\lambda_2 = \frac{y}{5}$$

$$\lambda_1 = z + \lambda_2 = z + \frac{y}{5}$$

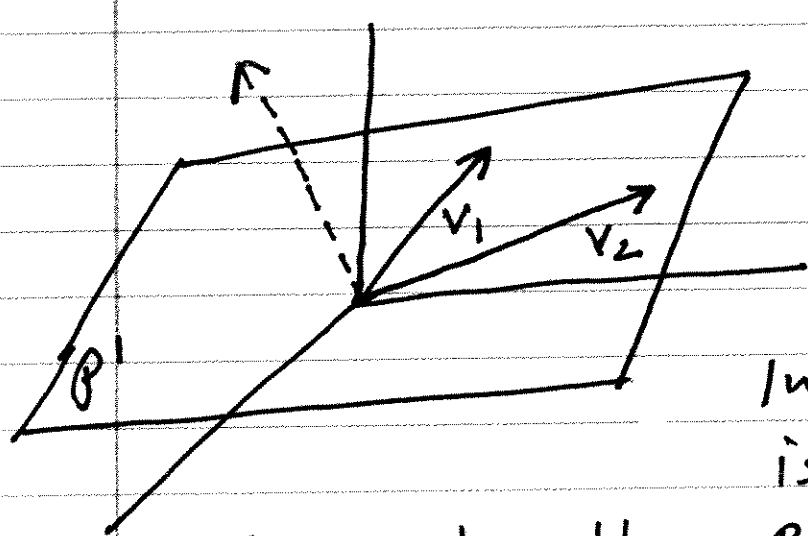
$$x = \lambda_1 + 3\lambda_2$$

$$= z + \frac{y}{5} + \frac{3}{5}y$$

$$= z + \frac{4}{5}y$$

$$\Rightarrow \boxed{5x - 4y - 5z = 0}$$

Eq^n of $P'$.




The coefficient vector $(5 \ -4 \ -5)$ is perpendicular to the vectors $v_1$ & $v_2$. In fact $(5 \ -4 \ -5)$ is perp. to every vector on the plane $P'$

## Example 10:

Let

$$u = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$$

be any vector in $\mathbb{R}^3$. Verify if $u \in P^1$ where $P^1$ is defined in Example 9.

Sol$^n$: If $u \in P^1$ we should be able to solve

$$\lambda_1 + 3\lambda_2 = 1$$
$$5\lambda_2 = 3 \qquad \text{for some } \lambda_1, \lambda_2.$$
$$\lambda_1 - \lambda_2 = -1$$

Let us try. — $\lambda_2 = 3/5$, $\lambda_1 = \lambda_2 - 1$.

$$= -\frac{2}{5}$$

$$\lambda_1 + 3\lambda_2 = -\frac{2}{5} + \frac{9}{5} = 7/5 \neq 1.$$

Hence we do not have a $sol^{\underline{n}}$ for the three equations.

Upshot:

If $u = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ $v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$

then

$$P = \{u + \lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \in \mathbb{R}\}.$$

is an affine 2-plane in $\mathbb{R}^3$.

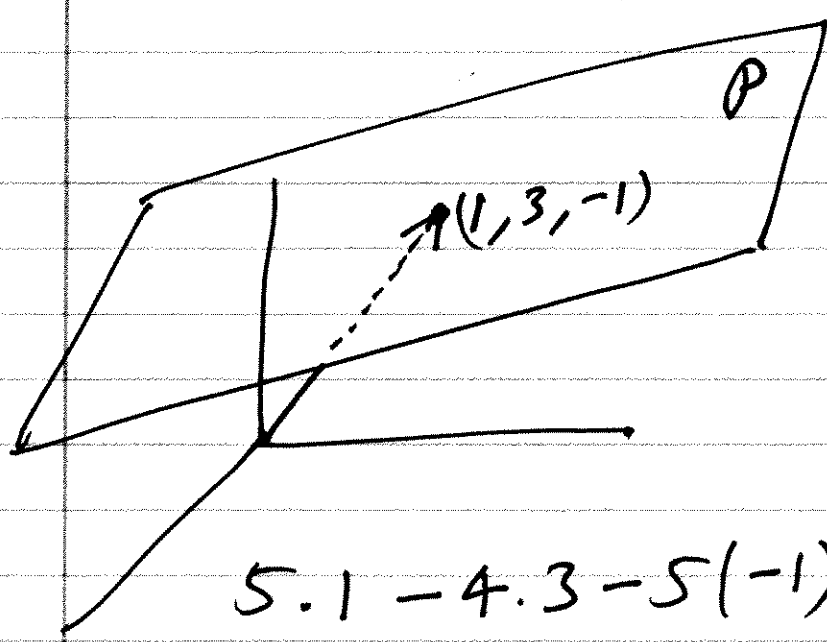
as opposed to homogeneous.

In fact $P$ is a parallel translation of the homogeneous plane in Ex 9.

To calculate the equation of $P$ we write

$$5x - 4y - 5z = d.$$

and find '$d$' such that the plane passes through $(1\ \ 3\ \ -1)$.




$$5.1 - 4.3 - 5(-1)$$

$$= 5 - 12 + 5 = -2.$$

$\therefore d = -2.$

$$\boxed{5x - 4y - 5z + 2 = 0}$$

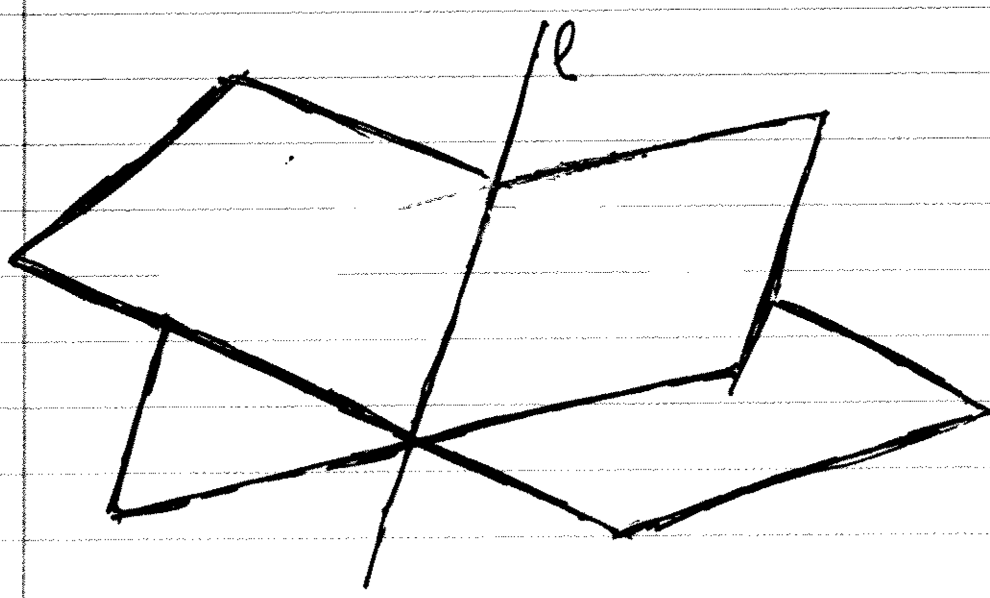
## Line Defined as an intersection of 2 planes in $\mathbb{R}^3$.

Consider the affine plane

$$5x-4y-5z+2=0$$

on page 28. We consider another plane given by equation

$$x-2y+z-3=0$$



The intersection of the two planes is an affine line $l$ in $\mathbb{R}^3$ (see fig).

To find a parametrization of $l$ we proceed as follows

$$\boxed{\begin{aligned} 5x - 4y - 5z + 2 &= 0 \\ x - 2y + z - 3 &= 0 \end{aligned}}$$

$\Downarrow$ 1 first eq$^n$ − 5 second eq$^n$. 3 second eq$^n$

$$\boxed{\begin{aligned} 0x + 6y - 10z + 17 &= 0 \\ 3x - 6y + 3z - 9 &= 0 \end{aligned}}$$

$\Downarrow$ same as 1st eq$^n$. 2nd eq$^n$ + 1st eq$^n$.

$$\boxed{\begin{aligned} 6y - 10z + 17 &= 0 \\ 3x - 7z + 8 &= 0 \end{aligned}}$$

$\Downarrow$

$$\boxed{z = \lambda,\ x = \frac{7}{3}\lambda - \frac{8}{3},\ y = \frac{5}{3}\lambda - \frac{17}{6}.}$$

Sol$^n$ parameterized by one parameter.

$\ell$ is described as follows:

$$\left\{ \begin{pmatrix} \frac{7}{3}\lambda - \frac{8}{3} \\ \frac{5}{3}\lambda - \frac{17}{6} \\ \lambda \end{pmatrix} : \lambda \in \mathbb{R} \right\}$$

If $u = \begin{pmatrix} -8/3 \\ -17/6 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 7/3 \\ 5/3 \\ 1 \end{pmatrix}$

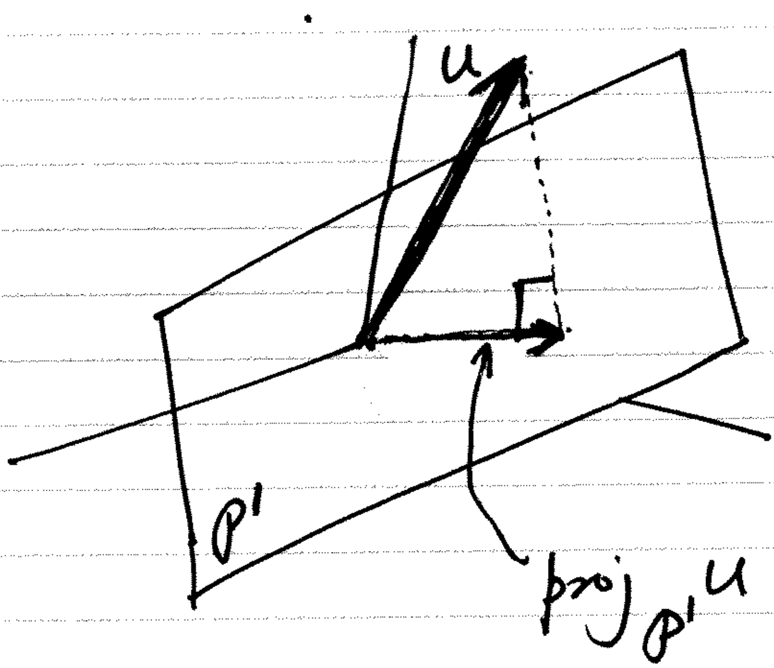
Line $\ell$ is in the familiar form.

$$\{u + \lambda v : \lambda \in \mathbb{R}\}$$

Cartesian Eqn of the line $l$ is given by

$$\boxed{\frac{x+8/3}{7/3} = \frac{y+17/6}{5/3} = \frac{3}{1}.}$$

## Projection of a vector on a homogeneous plane :—




Let $P'$ be a homogeneous plane defined as span of two vectors $v_1$ & $v_2$

$$P' = [v_1 \;\; v_2]$$

Let $u$ be a vector not in $P'$

$$u \notin P'.$$

We define

$$w = \mathrm{proj}_{P^1} u$$

as the orthogonal projection of $u$ on $P^1$ as follows:

$$u - w \perp P^1$$
$$w \in P^1$$

To calculate $w$ we proceed as follows:

- write $w = \lambda_1 v_1 + \lambda_2 v_2$.
- write $u - w = u - \lambda_1 v_1 - \lambda_2 v_2$
- $u - w \perp P^1 \Rightarrow \begin{matrix} u - w \perp v_1 \\ u - w \perp v_2 \end{matrix}$

$$\Downarrow$$

$$(u - \lambda_1 v_1 - \lambda_2 v_2) \cdot v_1 = 0$$
$$(u - \lambda_1 v_1 - \lambda_2 v_2) \cdot v_2 = 0$$

Hence

$$\boxed{\begin{aligned} \lambda_1(v_1 \cdot v_1) + \lambda_2(v_2 \cdot v_1) &= u \cdot v_1 \\ \lambda_1(v_1 \cdot v_2) + \lambda_2(v_2 \cdot v_2) &= u \cdot v_2 \end{aligned}}$$

Solving for $\lambda_1$ & $\lambda_2$ we obtain

$$\lambda_1 = \frac{(u \cdot v_1)\,\|v_2\|^2 - (u \cdot v_2)(v_1 \cdot v_2)}{\|v_1\|^2\,\|v_2\|^2 - (v_1 \cdot v_2)^2}$$

$$\lambda_2 = \frac{(u \cdot v_2)\,\|v_1\|^2 - (u \cdot v_1)(v_1 \cdot v_2)}{\|v_1\|^2\,\|v_2\|^2 - (v_1 \cdot v_2)^2}$$

Let $\alpha_1$ be the angle between $u$ & $v_1$

$\alpha_2$ " " " " $u$ & $v_2$

$\theta$ " " " " $v_1$ & $v_2$

we obtain

$$\lambda_1 = \frac{\|u\|}{\|v_1\|} \frac{\cos\alpha_1 - \cos\alpha_2 \cos\theta}{\sin^2\theta}$$

$$\lambda_2 = \frac{\|u\|}{\|v_2\|} \frac{\cos\alpha_2 - \cos\alpha_1 \cos\theta}{\sin^2\theta}$$

Substituting back we obtain

$$\omega = \lambda_1 v_1 + \lambda_2 v_2$$

$$= \frac{\|u\|}{\sin^2\theta}\left[(\cos\alpha_1 - \cos\alpha_2\cos\theta)\frac{v_1}{\|v_1\|} + (\cos\alpha_2 - \cos\alpha_1\cos\theta)\frac{v_2}{\|v_2\|}\right]$$

$$= \frac{\|u\|}{\sin^2\theta}\left[\cos\alpha_1\left(\frac{v_1}{\|v_1\|} - \cos\theta\frac{v_2}{\|v_2\|}\right) + \cos\alpha_2\left(\frac{v_2}{\|v_2\|} - \cos\theta\frac{v_1}{\|v_1\|}\right)\right.$$

Let

$$e_1 = \left[\frac{v_1}{\|v_1\|} - \cos\theta \frac{v_2}{\|v_2\|}\right] \Big/ \sin^2\theta$$

$$e_2 = \left[\frac{v_2}{\|v_2\|} - \cos\theta \frac{v_1}{\|v_1\|}\right] \Big/ \sin^2\theta$$

we have

$$w = \|u\| \left[\cos\alpha_1 \, e_1 + \cos\alpha_2 \, e_2\right]$$

<u>Sp case</u>

If we have $\theta = 90^\circ$ we get

$\cos\theta = 0 \quad \theta = 90$

$\sin\theta = 1 \quad \theta = 90$

Hence $e_1 = \dfrac{v_1}{\|v_1\|} \quad e_2 = \dfrac{v_2}{\|v_2\|}$.

$$\therefore w = \|u\| \left[\cos\alpha_1 \frac{v_1}{\|v_1\|} + \cos\alpha_2 \frac{v_2}{\|v_2\|}\right]$$

Thus if $e_1$ & $e_2$ are two orthonormal vectors in $\mathbb{R}^3$ then.

$$\boxed{\operatorname{proj}_{\underbrace{[e_1,e_2]}_{\omega}} u = \underbrace{(u\cdot e_1)}_{\|u\|\cos\alpha_1}\, \underbrace{e_1}_{\frac{v_1}{\|v_1\|}} + \underbrace{(u\cdot e_2)}_{\|u\|\cos\alpha_2}\, \underbrace{e_2}_{\frac{v_2}{\|v_2\|}}}$$

# A little linear algebra:

Let $V$ be a vector space

$V$ could be $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^{95}$

Let $v_1, v_2, \ldots, v_m$ be a set of $m$ vectors in $V$.

Def: We say that the set of vectors $v_1, \ldots, v_m$ are linearly independent (l.i.) if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

$$\Updownarrow$$

$$\alpha_1 = \alpha_2 = \cdots = \alpha_m = 0$$

A set of vectors is l.i. if the only linear combination which is zero is the trivial linear combination.

Conversely:

A set of vectors $v_1, \dots, v_m$ Is linearly dependent if there is at least one nontrivial l.c.

$$\beta_1 v_1 + \beta_2 v_2 + \dots + \beta_m v_m = 0$$

where $\beta_1, \dots, \beta_m$ are not all zeros.

Without any loss of generality assume that $\beta_1 \neq 0$. It follows that

$$v_1 = -\frac{\beta_2}{\beta_1} v_2 - \frac{\beta_3}{\beta_1} v_3 - \cdots - \frac{\beta_m}{\beta_1} v_m$$

Thus we say the following:

A set of vectors $v_1, \ldots, v_m$ is linearly dependent if we can write one vector as a l.c. of the other vectors.

Example 11:

Consider the following triplet of vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix}$$

in $\mathbb{R}^3$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \left.\begin{aligned} \alpha_1 + 2\alpha_2 - \alpha_3 &= 0 \\ \alpha_2 &= 0 \\ \alpha_1 + 3\alpha_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha_1 &= \alpha_3 \\ \alpha_2 &= 0 \\ \alpha_1 &= -3\alpha_3 \end{aligned}$$

$$\Downarrow$$

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 .$$

The vectors $v_1, v_2, v_3$ are l. i.

## Example 12

consider

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

in $\mathbb{R}^3$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\Rightarrow \left.\begin{aligned} \alpha_1 + 2\alpha_2 - \alpha_3 &= 0 \\ 2\alpha_1 + 4\alpha_2 - 2\alpha_3 &= 0 \\ -\alpha_1 - 2\alpha_2 + \alpha_3 &= 0 \end{aligned}\right\} \Rightarrow \alpha_1 + 2\alpha_2 - \alpha_3 = 0$$

All the other equations are redundant.

$$\Rightarrow \alpha_3 = \alpha_1 + 2\alpha_2 .$$

Choose

$\alpha_1 = 1 \quad \alpha_2 = 1$

we get $\alpha_3 = 3$

Hence

$V_1 + V_2 + 3V_3 = 0 \Rightarrow V_1, V_2, V_3$ are l.d.

Actually $V_2 = 2V_1$

$V_3 = -V_1$

Hence the vectors are l.d.

Why study linear independence??

We already know that

$P = [v_1, \dots, v_m]$ is a homogeneous m-plane in $V$, spanned by the vectors $v_1, \dots, v_m$.

If $v_1, \dots, v_m$ are l.i. then $P$ is always generated by at least $m$ vectors. This number $m$ is called the dimension of $P$.

## Example 11 (continued)

Using $v_1, v_2, v_3$ in Example 11 we can define a whole bunch of homogeneous planes.

$P_{12} = [v_1 \ v_2]$

$P_{13} = [v_1 \ v_3]$ Each of these planes have dimension 2.

$P_{23} = [v_2 \ v_3]$

We can also define three homogeneous lines

$l_1 = [v_1]$ $\quad l_2 = [v_2]$ $\quad l_3 = [v_3]$.

They all have dimension 1.

We can also define

$[v_1, v_2, v_3]$

It is homogeneous.
It has dimension 3
It is a hyperplane

Homogeneous 3 dimensional hyperplane in $\mathbb{R}^3$ is $\mathbb{R}^3$ itself.

Hence $\mathbb{R}^3 = [v_1\ v_2\ v_3]$.

Example 12 (continued)

Using $v_1, v_2, v_3$ is example 12 we define

$$l_1 = [v_1] \quad l_2 = [v_2] \quad l_3 = [v_3].$$

But all these lines are the same because

$$[v_2] = [2v_1] = [v_1]$$

$$[v_3] = [-v_1] = [v_1]$$

$$\therefore \; l = [v_1] = [v_2] = [v_3].$$

$$P_{12} = [v_1, v_2]$$

$$= [v_1, 2v_1] = [v_1]$$

Likewise $P_{13} = P_{23} = [v_1]$.

In particular

$P_{ij} = [v_i, v_j]$ is not 2-dimensional
but one dimensional.

$[v_1, v_2, v_3]$ is also one dimensional

$$[v_1 \; v_2 \; v_3] = [v_1].$$